Example $\displaystyle\sum_{k=1}^{n}\left(\left(4-3k\overset{-3k^2-18k}{(k+6)}\right)+k^3\right)$

$$=\sum_{k=1}^{n}\left(4-3k^2-18k+k^3\right)$$

$$=\sum_{k=1}^{n}4-3\sum_{k=1}^{n}k^2-18\sum_{k=1}^{n}k+\sum_{k=1}^{n}k^3$$

$$=\boxed{4\cdot n-3\frac{n(n+1)(2n+1)}{6}-18\frac{n(n+1)}{2}+\frac{n^2(n+1)^2}{4}}$$

Example Using the FUNdamental Theorem of Calculus, find $\displaystyle\int_1^2\frac{1}{x}\,dx$.

Solution: Since $(\ln(x))'=\frac{1}{x}$, then

$$\int_1^2\frac{1}{x}\,dx=\ln(x)\Big|_1^2=\ln(2)-\ln(1)$$

$$=\boxed{\ln 2}.$$

Similarly, $\displaystyle\int_1^B\frac{1}{x}\,dx=\ln(x)\Big|_1^B=\ln(B)-\ln(1)=\boxed{\ln(B)}.$

(This is how $\ln(B)$ is actually calculated.

Ex $\displaystyle\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$

Can we find $f(x)$ s.t.

$f'(x) = x\sin(x^2)$?

$$\left(-\frac{1}{2}\cos(x^2)\right)' = -\frac{1}{2}\left(-\sin(x^2)\cdot 2x\right) = x\sin(x^2).$$

$$\underset{\text{FTC}}{=} -\frac{1}{2}\cos(x^2)\Big|_0^{\sqrt{\pi}} = -\frac{1}{2}\cos\left(\left(\sqrt{\pi}\right)^2\right) - \left(-\frac{1}{2}\cos\left(0^2\right)\right)$$

$$= -\frac{1}{2}\cos(\pi) + \frac{1}{2}\cos(0)$$

$$= -\frac{1}{2}(-1) + \frac{1}{2}1 = \boxed{1}.$$

This is the same as

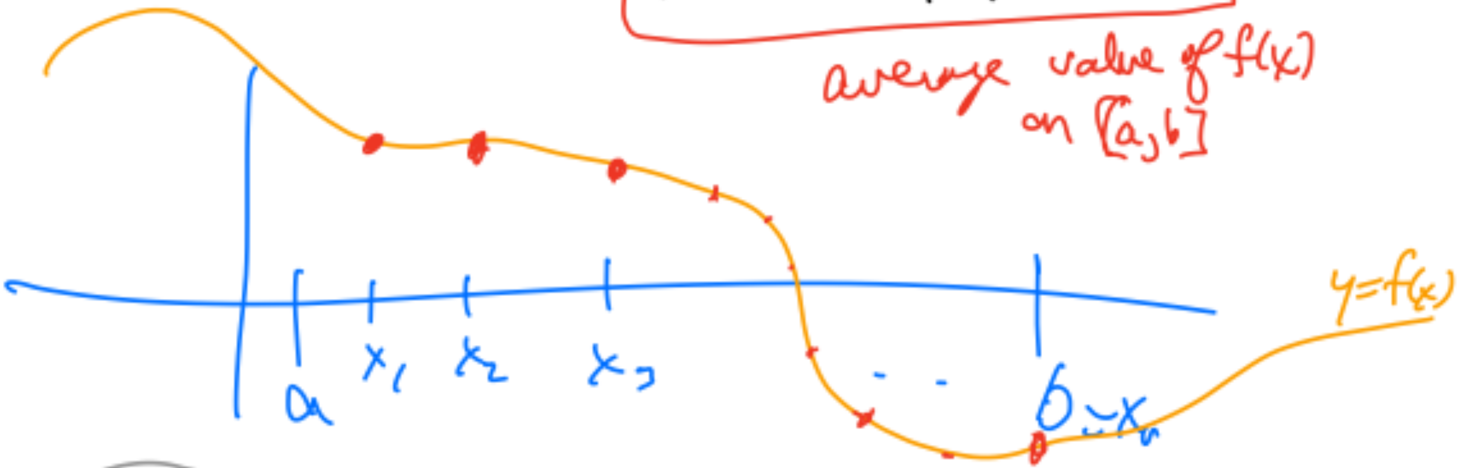
$$\lim_{n\to\infty}\sum_{k=1}^{n} \frac{k\sqrt{\pi}}{n}\sin\left(\frac{k^2\pi}{n^2}\right)\cdot\frac{\sqrt{\pi}}{n}$$

$\Delta x = \frac{\sqrt{\pi}}{n}$, $x_k = \frac{k\sqrt{\pi}}{n}$

Another interpretation of integrals:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \qquad \Delta x = \frac{(b-a)}{n}$$

$$= (b-a)\underbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k)}_{\text{average value of } f(x) \text{ on } [a,b]}.$$



$$\boxed{\int_a^b f(x)\,dx = (b-a)\cdot\left(\begin{array}{c}\text{average value of } f(x)\\ \text{on the interval}\\ [a,b].\end{array}\right)}$$

$$\boxed{\therefore \left(\begin{array}{c}\text{average value}\\ \text{of } f(x) \text{ on}\\ [a,b]\end{array}\right) = \frac{1}{(b-a)}\int_a^b f(x)\,dx.}$$

Because of the FTC, it's useful to find antiderivatives of fcns.

Notation for antiderivative:

$$\int f(x)\,dx = F(x)$$

general antiderivative of $f(x)$.

examples

$$\int e^x\,dx = e^x + C$$

any constant.

because $(e^x + C)' = e^x$.

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \cos(2x)\,dx = \frac{\sin(2x)}{2} + C$$

$$\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C \\ \ln(x) + C \end{cases}$$

$$\int x^7\,dx = \frac{1}{8}x^8 + C$$

$$\left[\ln(-x)\right]' = \frac{1}{-x}\cdot(-1) = \frac{1}{x}.$$ works if $x<0$.

$$\left[\ln(x)\right]' = \frac{1}{x}$$ ← works if $x>0$

To kill two birds with one stone

$$(\ln|x|)' = \frac{1}{x} \quad \text{if } x \neq 0.$$

Power Rule for antiderivatives:

$$\boxed{\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C & \text{if } n \neq -1 \\ \ln|x| + C & \text{if } n = -1. \end{cases}}$$

$\boxed{\text{Ex}}$ $$\int \frac{1}{\cos^2(x)}\,dx = \int \sec^2(x)\,dx = \tan(x) + C$$

$$\int_0^{\pi/6} \frac{1}{\cos^2(x)}\,dx = \tan(x)\Big|_0^{\pi/6}$$

$$= \tan\left(\frac{\pi}{6}\right) - \tan(0)$$

$$= \frac{1}{\sqrt{3}} - 0 = \boxed{\frac{1}{\sqrt{3}}}.$$

Ex $$\int_0^1 \frac{1}{\sqrt{1-t^2}}\,dt$$

$$= \arcsin(t)\Big|_0^1$$

$$= \arcsin(1) - \arcsin(0)$$

$$= \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}.$$

2^{nd} Fundamental Theorem of Calculus.

If f is continuous on the interval $[a,b]$, and $x \in [a,b]$, then if we define

$$g(x) = \int_a^x f(t)\,dt,$$

then $g'(x) = f(x)$.

Makes sense

antiderivative

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

$$\left(\int_a^x f(t)\,dt\right)' = \left(F(x) - F(a)\right)'$$

$$= F'(x) = f(x).$$

This also gives a way to construct an antiderivative!

That is.

$$g(x) = \int_a^x f(t)\,dt \quad \text{is}$$

an antiderivative of $f(x)$.

Sage math:

integral(f(x), x, a, b)

$$\left(\int_a^b f(x)\,dx \right)$$